A PRIORI BOUNDS FOR THE VORTICITY OF AXIS SYMMETRIC SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Abstract. We obtain a pointwise, a priori bound for the vorticity of axis symmetric solutions to the 3 dimensional Navier-Stokes equations. The bound is in the form of a reciprocal of a power of the distance to the axis of symmetry. This seems to be the first general pointwise estimate established for the axis symmetric Navier-Stokes equations.

1. Introduction

Recall the incompressible Navier-Stokes equations given in Cartesian coordinates:

$$\Delta v - (v \cdot \nabla)v - \nabla p - \partial_t v = 0, \ \text{div } v = 0$$

where the velocity field is $v = (v_1(x,t), v_2(x,t), v_3(x,t)) : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ and $p = p(x,t) : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ is the pressure. When one converts the system to cylindrical coordinates r, θ, z with $(x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$ and considers only those solutions that are axis symmetric, then solutions are restricted to ones of the form:

$$v(x,t) = v_r(r,z,t)\overrightarrow{e_r} + v_\theta(r,z,t)\overrightarrow{e_\theta} + v_z(r,z,t)\overrightarrow{e_z}.$$

The components v_r, v_θ, v_z are all independent of the angle of rotation θ. Note $\overrightarrow{e_r}, \overrightarrow{e_\theta}, \overrightarrow{e_z}$ are the basis vectors for $\mathbb{R}^3$given by:

$$\overrightarrow{e_r} = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \ \overrightarrow{e_\theta} = \left(\frac{-x_2}{r}, \frac{x_1}{r}, 0\right), \ \overrightarrow{e_z} = (0,0,1).$$

Much had been accomplished along the lines of axis symmetric solutions including the long time existence and uniqueness of strong solutions if the space region is taken to be all of $\mathbb{R}^3$, the external force, if any, as well as the initial velocity v_0, are axis symmetric, and the rotational components, f_θ and $v_{0,\theta}$, are equal to zero. That is, the no swirl case is known, and has been since the late 1960's (see O. A. Ladyzhenskaya [9], M. R. Uchoviskii & B. I. Yudovich [13], and S. Leonardi, J. Malek, J. Necas, & M. Pokorny [10]). More recent activities, in the presence of swirl, include the results of C.-C. Chen, R. M. Strain, T.-P.Tsai, & H.-T. Yau in [2] & [3], where they prove a lower bound on the blow-up rate of axis symmetric solutions. Similar to these results, more can be found in the work by G. Koch, N. Nadirashvili, G. Seregin, & V. Sverak [8]; under natural assumptions they address the types singularities that can occur in solutions to the Navier-Stokes equations. See also the work by G. Seregin & V. Sverak [11]. Also in the presence of swirl, there is the paper by J. Neustupa & M. Pokorny [6], proving the regularity of one component (either v_r or v_θ) implies regularity of the other components of the solution. Also proving regularity is the work of Q. Jiu & Z. Xin [7] under an assumption of sufficiently small

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zero dimension scaled norms. We would also like to mention the regularity results of D. Chae & J. Lee [1] who also prove regularity results assuming finiteness of another certain zero dimensional integral. Lastly we mention the results of G. Tian & Z. Xin [12], who constructed a family of singular axis symmetric solutions with singular initial data, as well as that of T. Hou & C. Li [4] who found a special class of global smooth solutions. See also a recent extension: T. Hou, Z. Lei & C. Li [5].

In our paper, in essence, we prove an upper bound for the (possible) blow up rate of the vorticity of axis symmetric solutions to the 3 dimensional Navier-Stokes equations. We first state a well-known a priori bound for the rotational component of the velocity; a proof can be found in [1] Section 3 Proposition 1, for example. From this we prove an a priori bound on ω_θ, the rotational component of the curl, in regions close to the axis of symmetry, using a Moser's Iteration argument similar to that found in the publication [14], as well as methods in [2]. With our bound on ω_θ, we derive a bound on the remaining components of the curl.

We state the theorem of the paper:

Theorem 1.1. *Suppose v is a smooth, axis symmetric solution of the 3 dimensional Navier-Stokes equations in $\mathbb{R}^3 \times (-T, 0)$ with initial data $v_0 = v(\cdot, -T) \in L^2(\mathbb{R}^3)$, and w is the vorticity. Assume further, $rv_{0,\theta} \in L^\infty(\mathbb{R}^3)$ and let $0 < R \le 1$. Then, there exist constants, B_1 and B_2, depending only on the initial data, such that for all $(x,t) \in P_{2,3,R} \subset \mathbb{R}^3 \times (-T, 0)$, where*
$P_{2,3,R} = \left\{(x,t) : 2R < \sqrt{x_1^2 + x_2^2} < 3R,\ -3R < x_3 < 3R,\ -R^2 < t < 0\right\}$*:*

(i) $|\omega_\theta(x,t)| \le \dfrac{B_1}{(x_1^2 + x_2^2)^{\frac{5}{2}}}$;

(ii) $|\omega_r(x,t)| + |\omega_z(x,t)| \le \dfrac{B_2}{(x_1^2 + x_2^2)^5}$.

Let us introduce some notation. We use $x = (x_1, x_2, x_3)$ to denote a point in $\mathbb{R}^3$ for rectangular coordinates, and in the cylindrical system we use $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \tan^{-1} \frac{x_2}{x_1}$, $z = x_3$. Let $R > 0$, $0 < A < B$ be constants, and define $P_{A,B,R}$ to be the region:

$$P_{A,B,R} = C_{A,B,R} \times (-R^2, 0)$$

where:

$$C_{A,B,R} = \{(x_1, x_2, x_3)|\ AR \le r \le BR,\ 0 \le \theta \le 2\pi,\ |z| \le BR\} \subset \mathbb{R}^3,$$

is the hollowed out cylinder centered at the origin, with inner radius AR, outer radius BR, and height extending up and down BR units for a total height of $2BR$.

Remark 1.1. *The constants B_1, B_2 in the above theorem are recorded here:*

$$B_1 = c\left(\|b\|^2_{L^\infty(-R^2,0;L^2(C_{1,4,R}))} + R\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}\right)^{\frac{5}{2}} \left(\|\omega_\theta\|_{L^2(P_{1,4,R})} + \sqrt{R}\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}\right),$$

$$B_2 = c\Bigg[\left(\|b\|^4_{L^\infty(-R^2,0;L^2(C_{\frac{1}{10},10,R}))} + R^2\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + R^2\right)\|\omega_\theta\|^2_{L^2(P_{\frac{1}{10},10,R})}$$
$$+R\|b\|^4_{L^\infty(-R^2,0;L^2(C_{\frac{1}{10},10,R}))} + \|v\|^2_{L^2(P_{\frac{1}{10},10,R})} + R^3\Bigg]^{\frac{5}{2}} \left(\|\omega_r\|_{L^2(P_{\frac{1}{10},10,R})} + \|\omega_z\|_{L^2(P_{\frac{1}{10},10,R})}\right),$$

where $b = (v_r, 0, v_z)$ and c is a generic constant. B_1 and B_2 depend only on the initial data, v_0, by standard energy estimates. Also they can be made to be independent of the smallness of R. Actually, $B_1, B_2 \to 0$ when $R \to 0$.

Remark 1.2. *We assume smoothness of the solution only for technical simplicity. One can use standard methods to treat the suitable weak solution case.*

The remainder of the paper is organized as follows:
Section 2: Preliminaries
Section 3: A priori bound for ω_θ
Section 4: A priori bound for ω_r and ω_z.

2. Preliminaries

Let us recall the standard conversion of the 3 dimensional axis symmetric Navier-Stokes equations to cylindrical form, (see [2] for example):

$$\begin{cases} \left(\Delta - \frac{1}{r^2}\right) v_r - (b \cdot \nabla) v_r + \frac{v_\theta^2}{r} - \frac{\partial p}{\partial r} - \frac{\partial v_r}{\partial t} = 0, \\ \left(\Delta - \frac{1}{r^2}\right) v_\theta - (b \cdot \nabla) v_\theta - \frac{v_\theta v_r}{r} - \frac{\partial v_\theta}{\partial t} = 0, \\ \Delta v_z - (b \cdot \nabla) v_z - \frac{\partial p}{\partial z} - \frac{\partial v_z}{\partial t} = 0, \\ \frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0, \end{cases}$$

where $b(x,t) = (v_r, 0, v_z)$ and the last equation is the divergence free condition. Here Δ represents the cylindrical scalar Laplacian and ∇ is the cylindrical gradient field which we record here:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad \nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right).$$

Notice, the equation for v_θ does not depend on the pressure. Defining $\Gamma = r v_\theta$, one sees that the function Γ satisfies:

$$\Delta \Gamma - (b \cdot \nabla)\Gamma - \frac{2}{r}\frac{\partial \Gamma}{\partial r} - \frac{\partial \Gamma}{\partial t} = 0, \text{ div } b = 0. \tag{2.1}$$

Also recall the vorticity field $\omega = \text{curl } v$ for axis symmetric solutions:

$$\omega(x,t) = \omega_r \vec{e_r} + \omega_\theta \vec{e_\theta} + \omega_z \vec{e_z},$$
$$\omega_r = -\frac{\partial v_\theta}{\partial z}, \ \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \ \omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}. \tag{2.2}$$

Next we record the equations of vorticity $\omega = \text{curl } v$, in cylindrical form (again, see [2] for example):

$$\begin{cases} \left(\Delta - \frac{1}{r^2}\right) \omega_r - (b \cdot \nabla)\omega_r + \omega_r \frac{\partial v_r}{\partial r} + \omega_z \frac{\partial v_r}{\partial z} - \frac{\partial \omega_r}{\partial t} = 0, \\ \left(\Delta - \frac{1}{r^2}\right) \omega_\theta - (b \cdot \nabla)\omega_\theta + 2\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial z} + \omega_\theta \frac{v_r}{r} - \frac{\partial \omega_\theta}{\partial t} = 0, \\ \Delta \omega_z - (b \cdot \nabla)\omega_z + \omega_z \frac{\partial v_z}{\partial z} + \omega_r \frac{\partial v_z}{\partial r} - \frac{\partial \omega_z}{\partial t} = 0. \end{cases}$$

Define $\Omega = \dfrac{\omega_\theta}{r}$, then we have that Ω satisfies:

$$\Delta\Omega - (b\cdot\nabla)\Omega + \frac{2}{r}\frac{\partial\Omega}{\partial r} - \frac{\partial\Omega}{\partial t} + \frac{2v_\theta}{r^2}\frac{\partial v_\theta}{\partial z} = 0, \text{ div } b = 0. \tag{2.3}$$

We confirm this by utilizing the fact that $r\Omega = \omega_\theta$ and thus satisfies the rotational equation for vorticity:

$$\left(\Delta - \frac{1}{r^2}\right)(r\Omega) - (b\cdot\nabla)(r\Omega) + \frac{2v_\theta}{r}\frac{\partial v_\theta}{\partial z} + \frac{v_r}{r}(r\Omega) - \frac{\partial(r\Omega)}{\partial t} = 0.$$

We compute with the product rule on each term:

$$\begin{aligned}
\Delta(r\Omega) =& r\frac{\partial^2\Omega}{\partial r^2} + 3\frac{\partial\Omega}{\partial r} + \frac{\Omega}{r} + r\frac{\partial^2\Omega}{\partial z^2},\\
-\frac{1}{r^2}(r\Omega) =& -\frac{\Omega}{r},\\
(-b\cdot\nabla)(r\Omega) =& -v_r\Omega - r(b\cdot\nabla)\Omega,\\
\frac{v_r}{r}(r\Omega) =& v_r\Omega,\\
-\frac{\partial}{\partial t}(r\Omega) =& -r\frac{\partial\Omega}{\partial t}.
\end{aligned}$$

We sum the above and the inhomogeneous term, $\dfrac{2v_\theta}{r}\dfrac{\partial v_\theta}{\partial z}$, to get:

$$r\frac{\partial^2\Omega}{\partial r^2} + \frac{\partial\Omega}{\partial r} + r\frac{\partial^2\Omega}{\partial z^2} - r(b\cdot\nabla)\Omega + 2\frac{\partial\Omega}{\partial r} - r\frac{\partial\Omega}{\partial t} + \frac{2v_\theta}{r}\frac{\partial v_\theta}{\partial z} = 0.$$

Grouping all but the last term, factoring out and dividing through by r, provides:

$$\Delta\Omega - (b\cdot\nabla)\Omega + \frac{2}{r}\frac{\partial\Omega}{\partial r} - \frac{\partial\Omega}{\partial t} + \frac{2v_\theta}{r^2}\frac{\partial v_\theta}{\partial z} = 0.$$

Notice equations (2.1) and (2.3) are similar except for a sign change on one term and the addition of an inhomogeneous term in (2.3). Equation (2.1) is used in [2] to provide the lower bound on the blow-up rate for axis symmetric solutions. As we work with equation (2.3) we assume the initial condition that provides for the pointwise bound of v_θ that appears in [1] which we restate below. Note, this is also implicitly stated in [6] (in Step 3.2 p. 396-397).

Proposition 2.1. *Suppose v is a smooth, axis symmetric solution of the 3 dimensional Navier-Stokes equations with initial data $v_0 \in L^2(\mathbb{R}^3)$. If $rv_{0,\theta} \in L^p(\mathbb{R}^3)$, then $rv_\theta \in L^\infty(0,T;L^p(\mathbb{R}^3))$. In particular, if $p=\infty$,*

$$|v_\theta(x,t)| \le \frac{||rv_{0,\theta}||_{L^\infty(\mathbb{R}^3)}}{\sqrt{x_1^2+x_2^2}}.$$

We will also utilize the scaling of the Navier-Stokes equations in conjunction with a change of variables. We recall that scaling of the equations now; the pair $(v(x,t),p(x,t))$ is a solution to the system, if and only if for any $k>0$ the re-scaled pair $(\widetilde{v}(x,t),\widetilde{p}(x,t))$ is also a solution, where

$$\widetilde{v}(x,t) = kv(kx,k^2t), \qquad \widetilde{p}(x,t) = k^2p(kx,k^2t).$$

Thus, if (v,p) is a solution to the axis symmetric Navier-Stokes equations for$(x,t) \in P_{1,4,k}$, then $(\widetilde{v}(\widetilde{x},\widetilde{t}), \widetilde{p}(\widetilde{x},\widetilde{t}))$ is a solution to the equation in the variables $\widetilde{x} = \frac{x}{k}, \widetilde{t} = \frac{t}{k^2}$ when $(\widetilde{x},\widetilde{t}) \in P_{1,4,1}$. We note here how certain quantities scale or change due to the above. Here D is any domain in $\mathbb{R}^3$ and $kD = \{x : x = ky,\ y \in D\}$:

$$r = \sqrt{x_1^2 + x_2^2} : \qquad \widetilde{r} = \sqrt{\left(\frac{x_1}{k}\right)^2 + \left(\frac{x_2}{k}\right)^2} = \frac{r}{k}$$

$$\|v(x,t)\|_{L^2(kD\times(-(kR)^2,0))} : \quad \begin{aligned} \|\widetilde{v}(\widetilde{x},\widetilde{t})\|_{L^2(D\times(-R^2,0))} &= \left(\int_{-R^2}^0 \int_D |\widetilde{v}(\widetilde{x},\widetilde{t})|^2 d\widetilde{x}d\widetilde{t}\right)^{\frac{1}{2}} \\ &= \left(\int_{-(kR)^2}^0 \int_{kD} |kv(x,t)|^2 \frac{1}{k^5} dxdt\right)^{\frac{1}{2}} \\ &= \frac{1}{k^{\frac{3}{2}}} \|v(x,t)\|_{L^2(kD\times(-(kR)^2,0))} \end{aligned}$$

$$b(x,t) = (v_r, 0, v_z) : \qquad \begin{aligned} \widetilde{b}(x,t) &= (kv_r(kx,k^2t), 0, kv_z(kx,k^2t)) \\ &= kb(kx,k^2t),\ (x,t) \in P_{1,4,k} \\ &\Rightarrow \widetilde{b}(\widetilde{x},\widetilde{t}) = kb(x,t) \end{aligned}$$

$$\|b(x,t)\|_{L^\infty(-(kR)^2,0;L^2(kD))} : \ \begin{aligned} \|\widetilde{b}(\widetilde{x},\widetilde{t})\|_{L^\infty(-R^2,0;L^2(D))} &= \sup_{-R^2 \le \widetilde{t} < 0} \left(\int_D |\widetilde{b}(\widetilde{x},\widetilde{t})|^2 d\widetilde{x}\right)^{\frac{1}{2}} \\ &= \sup_{-(kR)^2 \le t < 0} \left(\int_{kD} |kb(x,t)|^2 \frac{1}{k^3} dx\right)^{\frac{1}{2}} \\ &= \frac{1}{k^{\frac{1}{2}}} \|b(x,t)\|_{L^\infty(-(kR)^2,0;L^2(kD))} \end{aligned}$$

$$\omega(x,t) : \qquad \begin{aligned} \widetilde{\omega}(x,t) &= k^2\omega(kx,k^2t),\ (x,t) \in P_{1,4,k} \\ &\Rightarrow \widetilde{\omega}(\widetilde{x},\widetilde{t}) = k^2\omega(x,t) \end{aligned}$$

$$\|\omega(x,t)\|_{L^2(kD\times(-(kR)^2,0))} : \quad \begin{aligned} \|\widetilde{\omega}(\widetilde{x},\widetilde{t})\|_{L^2(D\times(-R^2,0))} &= \left(\int_{-R^2}^0 \int_D |\widetilde{\omega}(\widetilde{x},\widetilde{t})|^2 d\widetilde{x}d\widetilde{t}\right)^{\frac{1}{2}} \\ &= \left(\int_{-(kR)^2}^0 \int_{kD} |k^2\omega(x,t)|^2 \frac{1}{k^5} dxdt\right)^{\frac{1}{2}} \\ &= \frac{1}{k^{\frac{1}{2}}} \|\omega(x,t)\|_{L^2(kD\times(-(kR)^2,0))} \end{aligned}$$

One can show $\widetilde{\Gamma}(\widetilde{x},\widetilde{t}) = \widetilde{r}\widetilde{v_\theta}(\widetilde{x},\widetilde{t})$ is a solution to (2.1) and $\widetilde{\Omega}(\widetilde{x},\widetilde{t}) = \frac{\widetilde{\omega_\theta}(\widetilde{x},\widetilde{t})}{\widetilde{r}}$ is a solution to (2.3) in the variables $(\widetilde{x},\widetilde{t}) \in P_{1,4,1}$. We will do most of our computations on scaled cylinders.

3. A priori bound for ω_θ

In this Section, and in Section 4, we are going to drop the "tilde" notation for the sake of simplicity for a time when computations take place over the scaled cylinders. We will then recall that the $L^2 - L^\infty$ bounds derived are for scaled functions with a change of variables and we will discuss the consequences of this in subsections labeled "re-scaling". Note, however, because of this scaling, we must keep a close watch on constants that involve the quantities discussed in the preliminaries.

Proof of Theorem 1.1 (i):
In the region $P_{1,4,1}$ we do our analysis on (2.3):

$$\Delta\Omega - (b\cdot\nabla)\Omega + \frac{2}{r}\frac{\partial\Omega}{\partial r} - \frac{\partial\Omega}{\partial t} + \frac{2v_\theta}{r^2}\frac{\partial v_\theta}{\partial z} = 0, \ \text{div}\, b = 0.$$

A flow chart for the argument to prove part (i) of Theorem 1.1 is as follows:
Energy Estimates:
Step 1: Use a refined cut-off function.
Step 2: Estimate drift term $(b\cdot\nabla)\Omega$ using methods similar to [14].
Step 3: Estimate a term involving the cut-off.
Step 4: Estimate the term involving the directional derivative ∂_r using a method similar to that in [2].
Step 5: Estimate the inhomogeneous term utilizing the bound in Proposition 2.1 (see [1]).
$L^2 - L^\infty$ Estimate on Solutions to (2.3) via Moser's Iteration.
$L^2 - L^\infty$ Estimate on ω_θ via re-scaling.

Energy Estimates:

Step 1: We use a revised cut-off function and the equation to obtain inequality (3.4) below.

Let $q \geq 1$ be a rational number. We note that eventually we will be applying Moser's iteration, where at each step $q = \left(1+\frac{2}{n}\right)^i, \ i \in \mathbb{N}$ and here $n = 3$. Let

$$\Lambda = ||v_\theta||_{L^\infty(P_{1,4,1})} \leq ||rv_{0,\theta}||_{L^\infty(\mathbb{R}^3)} < \infty, \tag{3.1}$$

utilizing the hypothesis that $rv_{0,\theta} \in L^\infty(\mathbb{R}^3)$, the point-wise bound in Proposition 2.1, and the fact that $1 < \sqrt{x_1^2 + x_2^2} < 4$. Let

$$\overline{\Omega}_+(x,t) = \begin{cases} \Omega(x,t) + \Lambda & \Omega(x,t) \geq 0, \\ \Lambda & \Omega(x,t) < 0. \end{cases} \tag{3.2}$$

Note that $\overline{\Omega}_+ \geq \Lambda$ and all derivatives of $\overline{\Omega}_+$ on the set where $\Omega(x,t) < 0$ are equal to zero. This function is also Lipschitz and Ω we assume to be smooth. At interfaces boundary terms upon integration by parts will cancel and so the calculations below can be made

sense of. Direct computation yields:

$$\Delta\overline{\Omega}_+^q - (b\cdot\nabla)\overline{\Omega}_+^q + \frac{2}{r}\partial_r\overline{\Omega}_+^q - \partial_t\overline{\Omega}_+^q = -\frac{q\overline{\Omega}_+^{q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z} + q(q-1)\overline{\Omega}_+^{q-2}|\nabla\overline{\Omega}_+|^2. \tag{3.3}$$

Let $\frac{5}{8} \leq \sigma_2 < \sigma_1 \leq 1$. We let

$$P_{5-4\sigma_i,4\sigma_i,1} = \{(r,\theta,z)|\ (5-4\sigma_i) < r < 4\sigma_i,\ 0 \leq \theta \leq 2\pi,\ |z| < 4\sigma_i\} \times (-\sigma_i^2, 0)$$

for $i = 1,2$. For convenience denote the space portion, which is a hollowed out cylinder, as $C(\sigma_i)$ and let

$$P(\sigma_i) = P_{5-4\sigma_i,4\sigma_i,1} = C(\sigma_i) \times (-\sigma_i^2, 0).$$

Choose $\psi = \phi(y)\eta(s)$ to be a refined cut-off function satisfying:

$$\text{supp } \phi \subset C(\sigma_1);\ \phi(y) = 1 \text{ for all } y \in C(\sigma_2);\ \frac{|\nabla\phi|}{\phi^\delta} \leq \frac{c_1}{\sigma_1-\sigma_2} \text{ for } \delta \in (0,1);\ 0 \leq \phi \leq 1;$$

$$\text{supp } \eta \subset (-\sigma_1^2, 0];\ \eta(s) = 1, \text{ for all } s \in [-\sigma_2^2, 0];\ |\eta'| \leq \frac{c_2}{(\sigma_1-\sigma_2)^2};\ 0 \leq \eta \leq 1.$$

Let $f = \overline{\Omega}_+^q$ and use $f\psi^2$ as a test function in (3.3) to get:

$$\begin{aligned}
&\int_{P(\sigma_1)} (\Delta f - (b\cdot\nabla)f - \partial_s f + \frac{2}{r}\partial_r f) f\psi^2 dyds \\
&\qquad = \int_{P(\sigma_1)} q(q-1)\overline{\Omega}_+^{q-2}|\nabla\overline{\Omega}_+|^2 f\psi^2 dyds - \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z} f\psi^2 dyds \\
&\qquad = q(q-1)\int_{P(\sigma_1)} \overline{\Omega}_+^{-2}|\nabla\overline{\Omega}_+|^2 f^2\psi^2 dyds - \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2 dyds \\
&\qquad \geq -\int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2 dyds.
\end{aligned}$$

Integration by parts on the first term implies:

$$\begin{aligned}
&\int_{P(\sigma_1)} \nabla(f\psi^2)\nabla f dyds \\
&\qquad \leq \int_{P(\sigma_1)} \left(-b\cdot\nabla f(f\psi^2) - \partial_s f(f\psi^2) + \frac{2}{r}\partial_r f(f\psi^2) + \frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2\right) dyds
\end{aligned}$$

A manipulation using the product rule shows:

$$\int_{P(\sigma_1)} \nabla(f\psi^2)\nabla f dyds = \int_{P(\sigma_1)} \left(|\nabla(f\psi)|^2 - |\nabla\psi|^2 f^2\right) dyds.$$

Thus,

$$\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds \\
&\qquad \leq \int_{P(\sigma_1)} \left(-b\cdot\nabla f(f\psi^2) - \partial_s f(f\psi^2) + \frac{2}{r}\partial_r f(f\psi^2) + \frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2 + |\nabla\psi|^2 f^2\right) dyds.
\end{aligned}$$

Integration by parts on the term involving the time derivative yields:

$$\begin{aligned}\int_{P(\sigma_1)} -(\partial_s f) f\psi^2 dyds &= -\frac{1}{2}\int_{P(\sigma_1)} \partial_s(f^2)\psi^2 dyds \\ &= -\frac{1}{2}\left(\int_{C(\sigma_1)} f^2\psi^2(y,0)dy - \int_{C(\sigma_1)} f^2\psi^2(y,-\sigma_1^2)dy\right) \\ &\qquad + \frac{1}{2}\int_{P(\sigma_1)} \partial_s(\psi^2) f^2 dyds.\end{aligned}$$

Our cut-off functions provides $\psi^2 = (\phi\eta)^2$, $\eta(0) = 1$, $\eta(-\sigma_1^2) = 0$, and $0 \le \phi \le 1$. Thus,

$$\begin{aligned}\int_{P(\sigma_1)} -(\partial_s f) f\psi^2 dyds &= -\frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy + \int_{P(\sigma_1)} \phi^2(\eta\partial_s\eta) f^2 dyds \\ &\le -\frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy + \int_{P(\sigma_1)} (\eta\partial_s\eta) f^2 dyds.\end{aligned}$$

And so,

$$\begin{aligned}&\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\ &\qquad \le \int_{P(\sigma_1)} -b\cdot\nabla f(f\psi^2)dyds + \int_{P(\sigma_1)} (\eta\partial_s\eta + |\nabla\psi|^2) f^2 dyds \\ &\qquad\quad + \int_{P(\sigma_1)} \frac{2}{r}\partial_r f(f\psi^2)dyds + \int_{P(\sigma_1)} \frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2 dyds \\ &\qquad := T_1 + T_2 + T_3 + T_4.\end{aligned} \tag{3.4}$$

Step 2: To deal with T_1 we refer to [14] where a parabolic equation with a similar drift term is explored.

Since div $b = 0$,

$$\begin{aligned}T_1 &= \int_{P(\sigma_1)} -b\cdot(\nabla f)(f\psi^2)dyds \\ &= \frac{1}{2}\int_{P(\sigma_1)} -b\psi^2\cdot\nabla(f^2)dyds = \frac{1}{2}\int_{P(\sigma_1)} \text{div }(b\psi^2)f^2 dyds \\ &= \frac{1}{2}\int_{P(\sigma_1)} \text{div } b(\psi f)^2 dyds + \frac{1}{2}\int_{P(\sigma_1)} b\cdot\nabla(\psi^2)f^2 dyds \\ &= \int_{P(\sigma_1)} b\cdot(\nabla\psi)\psi f^2 dyds \\ &\le \left|\int_{P(\sigma_1)} \left(b\psi^{1+\delta}|f|^{2-a}\right)\left(\frac{\nabla\psi}{\psi^\delta}|f|^a\right) dyds\right|,\end{aligned}$$

for $0<\delta<1,\ 0<a<2$ which we introduce in order to split the above integral using Hölder's inequality. Apply Hölder's inequality with exponents $\frac{4}{3}$ and 4:

$$T_1 \leq \left(\int_{P(\sigma_1)} |b|^{\frac{4}{3}} \left(\psi^{1+\delta}|f|^{2-a}\right)^{\frac{4}{3}} dyds\right)^{\frac{3}{4}} \left(\int_{P(\sigma_1)} \left(\frac{|\nabla\psi|}{\psi^{\delta}}|f|^{a}\right)^{4} dyds\right)^{\frac{1}{4}}.$$

We would like $\frac{4}{3}(1+\delta)=2,\ \frac{4}{3}(2-a)=2$. This holds if $\delta=\frac{1}{2},\ a=\frac{1}{2}$. Using properties of the cutoff function we get:

$$T_1 \leq \left(\int_{P(\sigma_1)} |b|^{\frac{4}{3}}(f\psi)^2 dyds\right)^{\frac{3}{4}} \frac{c_1}{\sigma_1-\sigma_2}\left(\int_{P(\sigma_1)} f^2 dyds\right)^{\frac{1}{4}}.$$

Next we fix $\epsilon_1>0$ and we apply Young's inequality, with exponents $\frac{4}{3}$ and 4:

$$\begin{aligned}
T_1 &\leq \left(\frac{4}{3}\epsilon_1\right)^{\frac{3}{4}} \left(\int_{P(\sigma_1)} |b|^{\frac{4}{3}}(f\psi)^2 dyds\right)^{\frac{3}{4}} \cdot \left(\frac{4}{3}\epsilon_1\right)^{-\frac{3}{4}} \frac{c_1}{\sigma_1-\sigma_2}\left(\int_{P(\sigma_1)} f^2 dyds\right)^{\frac{1}{4}} \\
&\leq \epsilon_1 \int_{P(\sigma_1)} |b|^{\frac{4}{3}}(f\psi)^2 dyds + \frac{c_3\epsilon_1^{-3}}{(\sigma_1-\sigma_2)^4}\int_{P(\sigma_1)} f^2 dyds.
\end{aligned}$$

Thus,

$$|T_1| \leq \epsilon_1 c_4 K_b^{\frac{4}{3}}(C_{1,4,1})\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{c_3\epsilon_1^{-3}}{(\sigma_1-\sigma_2)^4}\int_{P(\sigma_1)} f^2 dyds, \tag{3.5}$$

where $K_b(C_{1,4,1})$ is the constant:

$$K_b(C_{1,4,1}) = \|b\|_{L^\infty(-1,0;L^2(C_{1,4,1}))}.$$

This last inequality holds as a result of $b=(v_r,0,v_z)\in L^\infty((0,\infty),L^2(\mathbb{R}^3))$, Hölder's inequality with exponents $\frac{3}{2}$ and 3, and the Sobolev Inequality, noting the dimension $n=3$:

$$\begin{aligned}
\int_{P(\sigma_1)} |b|^{\frac{4}{3}}(f\psi)^2 dyds &\leq \int_{-\sigma_1^2}^{0} \left(\int_{C(\sigma_1)} |b|^2 dy\right)^{\frac{2}{3}} \left(\int_{C(\sigma_1)} (f\psi)^6 dy\right)^{\frac{1}{3}} ds \\
&\leq c_4 \sup_{-\sigma_1^2\leq s\leq 0}\left(\int_{C(\sigma_1)} |b|^2 dy\right)^{\frac{2}{3}} \int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds \\
&\leq c_4 K_b^{\frac{4}{3}}(C_{1,4,1})\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds.
\end{aligned}$$

Step 3: The term involving the cut-off function, T_2, is standard. We use

$$T_2 = \int_{P(\sigma_1)} (\eta\partial_s\eta + |\nabla\psi|^2) f^2 dyds,$$

and properties of the cutoff,

$$|\nabla\psi|^2 = |\eta\nabla\phi|^2 \le \left(\frac{|\nabla\phi|}{\phi^\delta}\right)^2 \le \frac{c_1^2}{(\sigma_1-\sigma_2)^2}$$

and

$$|\eta\partial_s\eta| \le |\partial_s\eta| \le \frac{c_2}{(\sigma_1-\sigma_2)^2},$$

to get:

$$|T_2| \le \frac{c_5}{(\sigma_1-\sigma_2)^2}\int_{P(\sigma_1)} f^2 dyds. \tag{3.6}$$

Step 4: As we deal with $T_3 = \int_{P(\sigma_1)} \frac{2}{r}\partial_r f(f\psi^2)dyds$, we note we are assuming the integration takes place away from the singularity set of the solution to the axis symmetric Navier Stokes equations and away from the z-axis in general. Thus all functions are bounded and smooth and r varies between two positive constants, confirming this quantity is integrable. We also utilize the cylindrical coordinates of the axis symmetric case, and integration by parts:

$$\begin{aligned}
T_3 &= \int_{P(\sigma_1)} \frac{2}{r}\partial_r f(f\psi^2)dyds \\
&= \int_{P(\sigma_1)} \frac{1}{r}\partial_r(f^2)\psi^2 r dr d\theta dz ds \\
&= \int_{P(\sigma_1)} \partial_r(f^2)\psi^2 dr d\theta dz ds \\
&= -\int_{P(\sigma_1)} \partial_r(\psi^2) f^2 dr d\theta dz ds \\
&= -\int_{P(\sigma_1)} \frac{2}{r}\partial_r\psi(\psi f^2) r dr d\theta dz ds \\
&= -\int_{P(\sigma_1)} \frac{2}{r}\partial_r(\psi)(\psi f^2) dyds \\
&= -\int_{P(\sigma_1)} \frac{2}{r}\overrightarrow{e_r}\cdot\nabla\psi(\psi f^2)dyds.
\end{aligned}$$

The Cauchy-Schwartz inequality then implies:

$$|T_3| \le \int_{P(\sigma_1)} \frac{2}{r}|\nabla\psi|\psi f^2 dyds.$$

Next we use splitting methods similar to those found in [2]; fix $\epsilon_2 > 0,\ m > 1$ to be chosen later and apply Young's inequality with exponents m and $\frac{m}{m-1}$:

$$\begin{aligned}
|T_3| &\leq \int_{P(\sigma_1)} \frac{2}{r}|\nabla\psi|\psi f^2 dyds \\
&= \int_{P(\sigma_1)} \left((m\epsilon_2)^{\frac{1}{m}}\frac{2}{r}(\psi f)^{\frac{2}{m}}\right) \times \left((m\epsilon_2)^{\frac{-1}{m}}\psi^{\frac{m-2}{m}}|\nabla\psi| f^{\frac{2(m-1)}{m}}\right) dyds \\
&\leq \epsilon_2 \int_{P(\sigma_1)} \left(\frac{2}{r}\right)^m \psi^2 f^2 dyds + \frac{c_6 m^{\frac{m-2}{m-1}} \epsilon_2^{\frac{-1}{m-1}}}{m-1} \int_{P(\sigma_1)} \left(\frac{|\nabla\psi|}{\psi^{\frac{2-m}{m}}}\right)^{\frac{m}{m-1}} f^2 dyds.
\end{aligned}$$

Properties of the cutoff yield:

$$|T_3| \leq \epsilon_2 \int_{P(\sigma_1)} \left(\frac{2}{r}\right)^m (f\psi)^2 dyds + \frac{c_6 m^{\frac{m-2}{m-1}} \epsilon_2^{\frac{-1}{m-1}}}{(m-1)(\sigma_1-\sigma_2)^{\frac{m}{m-1}}} \int_{P(\sigma_1)} f^2 dyds.$$

Now consider the quantity:

$$\int_{C(\sigma_1)} \left(\frac{2}{r}\right)^m (f\psi)^2 dy.$$

Apply Hölder's inequality with exponents $\frac{3}{2}$ and $= 3$ and the Sobolev inequality, $n = 3$, then:

$$\begin{aligned}
\int_{C(\sigma_1)} \left(\frac{2}{r}\right)^m (f\psi)^2 dy &\leq \left(\int_{C(\sigma_1)} \left(\frac{2}{r}\right)^{\frac{3m}{2}} dy\right)^{\frac{2}{3}} \times \left(\int_{C(\sigma_1)} (f\psi)^6 dy\right)^{\frac{1}{3}} \\
&\leq c_7 \left(\int_{C(\sigma_1)} \left(\frac{2}{r}\right)^{\frac{3m}{2}} dy\right)^{\frac{2}{3}} \times \int_{P(\sigma_1)} |\nabla(f\psi)|^2 dy \\
&\leq c_{11} \int_{C(\sigma_1)} |\nabla(f\psi)|^2 dy,
\end{aligned}$$

if we choose m appropriately. To see this, we calculate:

$$\begin{aligned}
c_7 \left(\int_{C(\sigma_1)} \left(\frac{2}{r}\right)^{\frac{3m}{2}} dy\right)^{\frac{2}{3}} &= c_8 \left(\int_{-4\sigma_1}^{4\sigma_1} \int_0^{2\pi} \int_{\sigma_1}^{4\sigma_1} \frac{1}{r^{\frac{3m}{2}}} r dr d\theta dz\right)^{\frac{2}{3}} \\
&= \left(c_9 \sigma_1 \sigma_1^{-\frac{3m}{2}+2}\right)^{\frac{2}{3}} \qquad \text{if we choose } 1 < m < \frac{4}{3} \\
&= c_{10}(\sigma_1)^{2-m} \\
&\leq c_{11} \qquad \text{since } \frac{5}{8} \leq \sigma_2 < \sigma_1 \leq 1.
\end{aligned}$$

Note also:

$$\begin{aligned}
c_7\left(\int_{C(\sigma_1)}\left(\frac{2}{r}\right)^{\frac{3m}{2}}dy\right)^{\frac{2}{3}} &= c_8\left(\int_{-4\sigma_1}^{4\sigma_1}\int_0^{2\pi}\int_{\sigma_1}^{4\sigma_1}\frac{1}{r}drd\theta dz\right)^{\frac{2}{3}} && \text{if } m=\frac{4}{3}\\
&= c_9\sigma_1^{\frac{2}{3}}\\
&\le c_{10} && \text{since } \frac{5}{8}\le\sigma_2<\sigma_1\le 1.
\end{aligned}$$

Thus, allowing $1 < m \le \frac{4}{3}$ yields:

$$|T_3| \le \epsilon_2 c_{11}\int_{P(\sigma_1)}|\nabla(f\psi)|^2dyds + \frac{c_6 m^{\frac{m-2}{m-1}}\epsilon_2^{\frac{-1}{m-1}}}{(m-1)(\sigma_1-\sigma_2)^{\frac{m}{m-1}}}\int_{P(\sigma_1)}f^2dyds. \tag{3.7}$$

Step 5: Lastly we work on the inhomogeneous term of (2.3), that is $\frac{2v_\theta}{r^2}\frac{\partial v_\theta}{\partial z}$, which produced the term T_4. Recall $\Lambda = ||v_\theta||_{L^\infty(P_{1,4,1})} \le ||rv_{0,\theta}||_{L^\infty(\mathbb{R}^3)} < \infty$ and that $\overline{\Omega}_+ = \begin{cases} \Omega+\Lambda & \Omega\ge 0\\ \Lambda & \Omega<0\end{cases}$, thus $\overline{\Omega}_+ \ge \Lambda$. Also, we have let $f = \overline{\Omega}_+^q$. Using integration by parts yields:

$$\begin{aligned}
T_4 &= \int_{P(\sigma_1)}\frac{q\overline{\Omega}_+^{2q-1}}{r^2}\frac{\partial v_\theta^2}{\partial z}\psi^2dyds\\
&= -\int_{P(\sigma_1)}\frac{\partial}{\partial z}\left(\frac{\overline{\Omega}_+^{2q}\psi^2}{\overline{\Omega}_+}\right)\frac{q}{r^2}v_\theta^2dyds\\
&= -\int_{P(\sigma_1)}\frac{\partial}{\partial z}(f\psi)^2\frac{1}{\overline{\Omega}_+}\frac{q}{r^2}v_\theta^2dyds + \int_{P(\sigma_1)}(\overline{\Omega}_+^q\psi)^2\frac{1}{\overline{\Omega}_+^2}\frac{\partial\overline{\Omega}_+}{\partial z}\frac{q}{r^2}v_\theta^2dyds\\
&= -\int_{P(\sigma_1)}\frac{\partial}{\partial z}(f\psi)^2\frac{1}{\overline{\Omega}_+}\frac{q}{r^2}v_\theta^2dyds + \frac{1}{2}\int_{P(\sigma_1)}\frac{1}{\overline{\Omega}_+}\left[\frac{\partial(\overline{\Omega}_+^{2q}\psi^2)}{\partial z} - \overline{\Omega}_+^{2q}\frac{\partial\psi^2}{\partial z}\right]\frac{1}{r^2}v_\theta^2dyds\\
&= -\int_{P(\sigma_1)}\frac{\partial}{\partial z}(f\psi)^2\frac{1}{\overline{\Omega}_+}\frac{q-(1/2)}{r^2}v_\theta^2dyds - \frac{1}{2}\int_{P(\sigma_1)}\frac{1}{\overline{\Omega}_+}\overline{\Omega}_+^{2q}\frac{\partial\psi^2}{\partial z}\frac{1}{r^2}v_\theta^2dyds.
\end{aligned}$$

Considering $\frac{|v_\theta|}{\Lambda} \le 1$, utilizing $\Lambda \le \overline{\Omega}_+$, and $r = \sqrt{y_1^2+y_2^2} \ge 1$ for all $y \in P(\sigma_1)$, we continue by fixing $\epsilon_3 > 0$. Apply Young's inequality with exponents both being 2 to get:

$$\begin{aligned}
|T_4| &\le \int_{P(\sigma_1)}2q|v_\theta||f|\psi\left|\frac{\partial(f\psi)}{\partial z}\right|dyds + \frac{c_3}{\sigma_1-\sigma_2}\int_{P(\sigma_1)}f^2|v_\theta|dyds\\
&\le \int_{P(\sigma_1)}\left(\frac{2q\Lambda}{(2\epsilon_3)^{\frac{1}{2}}}f\psi\right)\times\left((2\epsilon_3)^{\frac{1}{2}}\frac{\partial(f\psi)}{\partial z}\right)dyds + \frac{c_3\Lambda}{\sigma_1-\sigma_2}\int_{P(\sigma_1)}f^2dyds \qquad (3.8)\\
&\le c_{12}\Lambda^2q^2\epsilon_3^{-1}\int_{P(\sigma_1)}f^2dyds + \epsilon_3\int_{P(\sigma_1)}|\nabla(f\psi)|^2dyds + \frac{c_3\Lambda}{\sigma_1-\sigma_2}\int_{P(\sigma_1)}f^2dyds.
\end{aligned}$$

$L^2 - L^\infty$ **Estimate:** An $L^2 - L^\infty$ bound is derived using Moser's iteration. Recall inequality (3.4) from Step 1 and substitute the estimates for T_1 (3.5), T_2 (3.6), T_3 (3.7), T_4 (3.8), found in Step 2-Step 5 to obtain:

$$\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\
&\quad\le \epsilon_1 c_4 K_b^{\frac{4}{3}}(C_{1,4,1}) \int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{c_3\epsilon_1^{-3}}{(\sigma_1-\sigma_2)^4}\int_{P(\sigma_1)} f^2 dyds \\
&\qquad + \frac{c_5}{(\sigma_1-\sigma_2)^2}\int_{P(\sigma_1)} f^2 dyds \\
&\qquad + \epsilon_2 c_{11}\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{c_6 m^{\frac{m-2}{m-1}}\epsilon_2^{\frac{-1}{m-1}}}{(m-1)(\sigma_1-\sigma_2)^{\frac{m}{m-1}}}\int_{P(\sigma_1)} f^2 dyds \\
&\qquad + \epsilon_3\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + c_{12}\Lambda^2 q^2\epsilon_3^{-1}\int_{P(\sigma_1)} f^2 dyds.
\end{aligned}$$

Choose

$$\epsilon_1 = \frac{1}{6c_4 K_b^{\frac{4}{3}}(C_{1,4,1})},\ \epsilon_2 = \frac{1}{6c_{11}},\ \epsilon_3 = \frac{1}{6}$$

and absorb the appropriate terms to the left hand side. Then, we have the following:

$$\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\
&\quad\le \left(\frac{c_{13}K_b^4(C_{1,4,1})}{(\sigma_1-\sigma_2)^4} + \frac{c_{14}}{(\sigma_1-\sigma_2)^2} + \frac{c_{15}m^{\frac{m-2}{m-1}}}{(m-1)(\sigma_1-\sigma_2)^{\frac{m}{m-1}}} + c_{16}q^2\Lambda^2\right)\int_{P(\sigma_1)} f^2 dyds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\
&\qquad\qquad \le \frac{c_{17}q^2}{(\sigma_1-\sigma_2)^4}\left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)\int_{P(\sigma_1)} f^2 dyds.
\end{aligned} \tag{3.9}$$

The last inequality follows with $q = 1 + \frac{2}{n} > 1$ and $0 < \sigma_1 - \sigma_2 < 1$, if m is such that $\frac{m}{m-1} \le 4$. This implies $m \ge \frac{4}{3}$, but our previous restriction on m required $1 < m \le \frac{4}{3}$. Thus, we let $m = \frac{4}{3}$ and deduce (3.9) above.

Moser's Iteration: We claim that Moser's iteration process and the estimate (3.9) together imply:

$$\sup_{P_{2,3,1}} \overline{\Omega}_+^2 \le c_{21}\left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}}\int_{P_{1,4,1}} \overline{\Omega}_+^2 dyds.$$

Hölder's inequality and the Sobolev inequality imply:

$$\int_{\mathbb{R}^n} (f\phi)^{2(1+\frac{2}{n})} dy \le \left(\int_{\mathbb{R}^n} (f\phi)^2 dy \right)^{\frac{2}{n}} \left(\int_{\mathbb{R}^n} (f\phi)^{\frac{2n}{n-2}} dy \right)^{\frac{n-2}{n}}$$
$$\le c_{18} \left(\int_{\mathbb{R}^n} (f\phi)^2 dy \right)^{\frac{2}{n}} \left(\int_{\mathbb{R}^n} |\nabla (f\phi)|^2 dy \right).$$

Multiply by the time portion of the cut-off function to the correct power, $\eta^{2(1+\frac{2}{n})}(s)$, on both sides and integrate over time; one can deduce:

$$\int_{-\sigma_1^2}^0 \int_{\mathbb{R}^n} (f\psi)^{2(1+\frac{2}{n})} dyds \le c_{18} \sup_{-\sigma_1^2 \le s \le 0} \left(\int_{\mathbb{R}^n} (f\psi)^2 dy \right)^{\frac{2}{n}} \int_{-\sigma_1^2}^0 \int_{\mathbb{R}^n} |\nabla (f\psi)|^2 dyds.$$

We use properties of the cut-off to obtain:

$$\int_{P(\sigma_1)} (\psi f)^{2(1+\frac{2}{n})} dyds \le c_{18} \left(\sup_{-\sigma_1^2 \le s < 0} \int_{C(\sigma_1)} (f\psi)^2 (y,s) dy \right)^{\frac{2}{n}} \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds. \tag{3.10}$$

In fact, with $n = 3$ the above is:

$$\int_{P(\sigma_1)} (\psi f)^{\frac{10}{3}} dyds \le c_{18} \left(\sup_{-\sigma_1^2 \le s < 0} \int_{C(\sigma_1)} (f\phi)^2 (y,s) dy \right)^{\frac{2}{3}} \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds. \tag{3.11}$$

We are noting this here because we will use this later in Section 4. The above argument can be run for each time level $-\sigma_1^2 \le s < 0$ and in fact (3.9) holds for all s in this interval as the upper time limit of the time cut-off function. Thus, the second to last factor on the right hand side of inequality (3.10) is still controlled by estimate (3.9). So together with the estimate and the cut-off function again, we get:

$$\int_{P(\sigma_2)} \overline{\Omega}_+^{2q\gamma} dyds \le c_{18} \left(\frac{c_{16} q^2}{\tau^4} \left(K_b^4 (C_{1,4,1}) + \Lambda^2 + 1 \right) \int_{P(\sigma_1)} \overline{\Omega}_+^{2q} dyds \right)^{\gamma}, \tag{3.12}$$

where $\gamma = 1 + \frac{2}{n}$, $\tau = \sigma_1 - \sigma_2$.

Let $\tau_i = 2^{-i-2}$, $\sigma_0 = 1$, $\sigma_i = \sigma_{i-1} - \tau_i = 1 - \sum_{j=1}^i \tau_j$, $q = \gamma^i$. Recall $P(\sigma_i) = P_{5-4\sigma_i, 4\sigma_i, 1}$. Then (3.12) generalizes to:

$$\int_{P(\sigma_{i+1})} \overline{\Omega}_+^{2\gamma^{i+1}} dyds \le c_{18} \left(c_{19}^{i+2} \gamma^{2i} \left(K_b^4 (C_{1,4,1}) + \Lambda^2 + 1 \right) \int_{P(\sigma_i)} \overline{\Omega}_+^{2\gamma^i} dyds \right)^{\gamma}, \tag{3.13}$$

which, after taking the $\frac{1}{\gamma}$-th power of both sides, implies:

$$\left(\int_{P(\sigma_{i+1})} \overline{\Omega}_+^{2\gamma^{i+1}} dyds \right)^{\frac{1}{\gamma}} \le c_{18}^{\frac{1}{\gamma}} \left(c_{19}^{i+2} \gamma^{2i} \left(K_b^4 (C_{1,4,1}) + \Lambda^2 + 1 \right) \int_{P(\sigma_i)} \overline{\Omega}_+^{2\gamma^i} dyds \right).$$

After iterating the above process, that is, using (3.13) on the integral on the left and raising both sides to the $\frac{1}{\gamma}$-th power repeatedly, one obtains:

$$\left(\int_{P(\sigma_{i+1})} \overline{\Omega}_+^{2\gamma^{i+1}} dyds\right)^{\gamma^{-i-1}}$$
$$\leq c_{18}^{\sum \gamma^{-j}} c_{19}^{\sum (j+1)\gamma^{-j+1}} \gamma^{2\sum (j-1)\gamma^{-j+1}} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\sum \gamma^{-j+1}} \int_{P_{1,4,1}} \overline{\Omega}_+^2 dyds.$$

Note the sums in the exponents are all from $j = 1$ to $j = i+1$. Let $i \to \infty$. All the exponent series converge. In particular, the series in the exponent for $\left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)$ converges to $\frac{5}{2}$. Note also that $\sigma_i \to \frac{3}{4}$, and so:

$$\sup_{P_{2,3,1}} \overline{\Omega}_+^2 \leq c_{20} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \int_{P_{1,4,1}} \overline{\Omega}_+^2 dyds. \tag{3.14}$$

Next, repeating the argument on $\overline{\Omega}_- = \begin{cases} -\Omega + \Lambda & \Omega \leq 0 \\ \Lambda & \Omega > 0 \end{cases}$ yields:

$$\sup_{P_{2,3,1}} \overline{\Omega}_-^2 \leq c_{20} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \int_{P_{1,4,1}} \overline{\Omega}_-^2 dyds.$$

Recall

$$\overline{\Omega}_+ = \begin{cases} \Omega + \Lambda & \Omega \geq 0 \\ \Lambda & \Omega < 0 \end{cases} \qquad \overline{\Omega}_- = \begin{cases} -\Omega + \Lambda & \Omega \leq 0 \\ \Lambda & \Omega > 0 \end{cases}$$

$$\Omega = \overline{\Omega}_+ - \overline{\Omega}_- \qquad \Lambda = \|v_\theta\|_{L^\infty(P_{1,4,1})} \leq \|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}$$

Thus,

$$\begin{aligned}
\sup_{P_{2,3,1}} \Omega^2 &\leq \sup_{P_{2,3,1}} \left(\overline{\Omega}_+ - \overline{\Omega}_-\right)^2 \\
&\leq c_{20} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \sup_{P_{2,3,1}} \left(\overline{\Omega}_+^2 + \overline{\Omega}_-^2\right) \\
&\leq c_{20} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \left(\int_{P_{1,4,1}} \overline{\Omega}_+^2 dyds + \int_{P_{1,4,1}} \overline{\Omega}_-^2 dyds\right) \\
&\leq c_{20} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \left(\int_{\{\Omega \geq 0\}} (\Omega + \Lambda)^2 dyds + \int_{\{\Omega < 0\}} \Lambda^2 dyds \right. \\
&\qquad \left. + \int_{\{\Omega \leq 0\}} (-\Omega + \Lambda)^2 dyds + \int_{\{\Omega > 0\}} \Lambda^2 dyds\right) \\
&\leq c_{20} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \left(\int_{P_{1,4,1}} (\Omega + \Lambda)^2 + (-\Omega + \Lambda)^2 + 2\Lambda^2 dyds\right) \\
&= c_{20} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \left(2\int_{P_{1,4,1}} \Omega^2 dyds + 4\int_{P_{1,4,1}} \Lambda^2 dyds\right) \\
&\leq c_{21} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right)^{\frac{5}{2}} \left(\|\Omega\|_{L^2(P_{1,4,1})}^2 + \Lambda^2\right).
\end{aligned}$$

Re-scaling: We now recall that we omitted the "tildes" in the notation in the above computations. So what has actually been proven thus far is:

$$\sup_{(\widetilde{x},\widetilde{t})\in P_{2,3,1}} \widetilde{\Omega}^2(\widetilde{x},\widetilde{t}) \le c_{21}\left(K^4_{\widetilde{b}}(C_{1,4,1})+\widetilde{\Lambda}^2+1\right)^{\frac{5}{2}}\left(\|\widetilde{\Omega}\|^2_{L^2(P_{1,4,1})}+\widetilde{\Lambda}^2\right).$$

Recall $\widetilde{x}=\frac{x}{k}$, $\widetilde{t}=\frac{t}{k^2}$, $\widetilde{\Omega}(\widetilde{x},\widetilde{t})=\frac{\widetilde{\omega_\theta}(\widetilde{x},\widetilde{t})}{\widetilde{r}}$. So with $2\le\widetilde{r}\le 3$ on the left and $1\le\widetilde{r}\le 4$ on the right we can derive:

$$\sup_{(\widetilde{x},\widetilde{t})\in P_{2,3,1}} \widetilde{\omega}^2_\theta(\widetilde{x},\widetilde{t}) \le c_{22}\left(K^4_{\widetilde{b}}(C_{1,4,1})+\widetilde{\Lambda}^2+1\right)^{\frac{5}{2}}\left(\int_{P_{1,4,1}}\widetilde{\omega}^2_\theta(\widetilde{x},\widetilde{t})d\widetilde{x}d\widetilde{t}+\widetilde{\Lambda}^2\right).$$

We recall from the Section 2 Preliminaries :

$$K_{\widetilde{b}}(C_{1,4,1})=\|\widetilde{b}(\widetilde{x},\widetilde{t})\|_{L^\infty(-1,0;L^2(C_{1,4,1}))}=\frac{1}{k^{\frac{1}{2}}}\|b(x,t)\|_{L^\infty(-k^2,0;L^2(C_{1,4,k}))}$$

and

$$\|\widetilde{\omega}(\widetilde{x},\widetilde{t})\|_{L^2(P_{1,4,1})}=\frac{1}{k^{\frac{1}{2}}}\|\omega(x,t)\|_{L^2(P_{1,4,k})}.$$

Also we note the control on Λ is a scaling invariant quantity. Since $\Lambda=\|v_\theta\|_{L^\infty(P_{1,4,1})}$, we use Proposition 2.1:

$$\begin{aligned}\widetilde{\Lambda}&=\left(\sup_{P_{1,4,1}}|\widetilde{v_\theta}(\widetilde{x},\widetilde{t})|\right)\\ &\le\left(\||\widetilde{r}\widetilde{v}_\theta(\widetilde{x},-T)\|_{L^\infty(\mathbb{R}^3)}\right) \qquad\text{applying Proposition 2.1,}\\ &=\|rv_\theta(x,-T)\|_{L^\infty(\mathbb{R}^3)}\\ &=\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}.\end{aligned}$$

We utilize $0<k<1$ to obtain:

$$\begin{aligned}&\sup_{(x,t)\in P_{2,3,k}} k^4\omega^2_\theta(x,t)\\ &\le c_{22}\left(\frac{1}{k^2}\|b\|^4_{L^\infty(-k^2,0;L^2(C_{1,4,k}))}+\|rv_{0,\theta}\|^2_{L^\infty(\mathbb{R}^3)}\right)^{\frac{5}{2}}\left(\int_{P_{1,4,k}}k^4\omega^2_\theta(x,t)\frac{1}{k^5}dxdt+\|rv_{0,\theta}\|^2_{L^\infty(\mathbb{R}^3)}\right)\\ &\le\frac{c_{23}}{k^6}\left(\|b\|^2_{L^\infty(-k^2,0;L^2(C_{1,4,k}))}+k\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}\right)^5\left(\|\omega_\theta\|^2_{L^2(P_{1,4,k})}+k\|rv_{0,\theta}\|^2_{L^\infty(\mathbb{R}^3)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\omega_\theta(x,t)\|_{L^\infty(P_{2,3,k})}\\ &\qquad\le\frac{c_{24}}{k^5}\left(\|b\|^2_{L^\infty(-k^2,0;L^2(C_{1,4,k}))}+k\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}\right)^{\frac{5}{2}}\left(\|\omega_\theta\|_{L^2(P_{1,4,k})}+\sqrt{k}\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}\right).\end{aligned}$$

This proves part (i) of Theorem 1.1.

Note, the way the cubes on the left and right are related is that on the right, we have $\frac{1}{2}$ of the inner radius and $\frac{4}{3}$ of the outer radius.

4. A priori bounds for ω_r and ω_z

In this section we use the a priori bound established in part (i) of Theorem 1.1 (ie. $|\omega_\theta| \le \frac{B_1}{r^5}$) and the 2×2 system below, which consists of the two remaining curl equations noted before, to derive a priori bounds for ω_r and ω_z.

$$\begin{cases} \Delta\omega_r - (b\cdot\nabla)\omega_r + \omega_r\left(\dfrac{\partial v_r}{\partial r} - \dfrac{1}{r^2}\right) + \omega_z\dfrac{\partial v_r}{\partial z} - \dfrac{\partial \omega_r}{\partial t} = 0, \\ \Delta\omega_z - (b\cdot\nabla)\omega_z + \omega_z\dfrac{\partial v_z}{\partial z} + \omega_r\dfrac{\partial v_z}{\partial r} - \dfrac{\partial \omega_z}{\partial t} = 0. \end{cases} \tag{4.1}$$

The drift term, $b\cdot\nabla$ can be dealt with in a similar manner to that in Section 3. The main work is to treat the potential terms where $\dfrac{\partial v_r}{\partial r} - \dfrac{1}{r^2}$, $\dfrac{\partial v_r}{\partial z}$, $\dfrac{\partial v_z}{\partial r}$, $\dfrac{\partial v_z}{\partial r}$ are regarded as potentials. It turns out one can control the $L^{\frac{10}{3}}$ norm of these using the a priori bound on ω_θ established in part (i) of Theorem 1.1 and the a priori bound on v_θ from Proposition 2.1. These $L^{\frac{10}{3}}$ bounds are sufficient to prove part (ii) of Theorem 1.1.

We need two lemmas which are localized versions of Lemma 2 and Lemma 3 in [6], and very similar, also, to Lemma 3 in [1]. Both should be known, but the proofs are short and are included here for completeness. First we recall our notation, $C_{A,B,R} = \{(r,\theta,z)|\ AR \le r \le BR,\ 0\le\theta\le 2\pi,\ |z|\le BR\} \subset \mathbb{R}^3$, and $P_{A,B,R} = C_{A,B,R}\times(-R^2,0)$.

Lemma 4.1. *Let $v \in C^\infty(C_{1,4,1})$ be a vector field. Then, for all $q>1$, there exists a constant, $c(q)>0$, such that*

$$\|\nabla v\|_{L^q(C_{2,3,1})} \le c(q)\left(\|curl\ v\|_{L^q(C_{1,4,1})} + \|div\ v\|_{L^q(C_{1,4,1})} + \|v\|_{L^q(C_{1,4,1})}\right).$$

Proof. Define ϕ to be a cut-off function such that $\phi\in C_0^\infty(C_{1,4,1})$, $0\le\phi\le 1$, $\phi = 1$ in $C_{2,3,1}$, $|\nabla\phi|\le c_1$, a constant. Then $v\phi$ is compactly supported, and it is well known that:

$$\|\nabla(v\phi)\|_{L^q(C_{1,4,1})} \le c(q)\left(\|\text{curl }(v\phi)\|_{L^q(C_{1,4,1})} + \|\text{div }(v\phi)\|_{L^q(C_{1,4,1})}\right). \tag{4.2}$$

(This is sometimes called the Helmholtz or Hodge decomposition). Next note

$$\begin{aligned} \text{div }(v\phi) &= \text{div } v\ \phi + v\cdot\nabla\phi \qquad \text{and} \\ \text{curl }(v\phi) &= \text{curl } v\ \phi + \nabla\phi\times v. \end{aligned}$$

The lemma follows by substituting the last two identities into the right hand side of (4.2) and using the Minkowski inequality and properties of the cutoff function. □

The following lemma is a generalization of Lemma 3 in [6].

Lemma 4.2. *Let $v = v(x,t)$ be a divergence free, axis symmetric, smooth vector field in $Q_{1,4} = C_{1,4,1}\times[-T,T]$ for fixed $T>0$. Then, for all $q>1$, there exists a constant, $c = c(q)>0$, such that*

$$\begin{aligned} &\|\nabla v_r\|_{L^q(Q_{2,3})} + \left\|\frac{v_r}{r}\right\|_{L^q(Q_{2,3})} + \|\nabla v_z\|_{L^q(Q_{2,3})} \\ &\qquad \le c(q)\left(\|(curl\ v)_\theta\|_{L^q(Q_{1,4})} + \|v\|_{L^q(Q_{1,4})}\right). \end{aligned}$$

Proof. In the cylindrical coordinate system, for an axis symmetric vector field, div $v = 0$ means

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0.$$

Therefore the vector field:

$$\overline{v} = v_r \overrightarrow{e_r} + v_z \overrightarrow{e_z}$$

is still divergence free. Since the inequality we want to prove does not involve v_θ, we first work on $\overline{v}$ where v_θ is not involved. Also $\overline{v}$ is axis symmetric, and so curl $\overline{v}$ has only one nonzero component, the one in the direction of $\overrightarrow{e_\theta}$. This is because for axis symmetric vector fields:

$$\omega(x,t) = \omega_r \overrightarrow{e_r} + \omega_\theta \overrightarrow{e_\theta} + \omega_z \overrightarrow{e_z}$$

$$\omega_r = -\frac{\partial v_\theta}{\partial z},\ \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r},\ \omega_z = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}.$$

Thus,

$$\text{curl } \overline{v} = (\text{curl } \overline{v})_\theta \overrightarrow{e_\theta}.$$

Applying Lemma 4.1 on $\overline{v}$, we deduce, for any fixed t:

$$\begin{aligned}\|\nabla \overline{v}(\cdot,t)\|_{L^q(C_{2,3,1})} &\le c(q)\left(\|\text{curl } \overline{v}(\cdot,t)\|_{L^q(C_{1,4,1})} + \|\overline{v}(\cdot,t)\|_{L^q(C_{1,4,1})}\right)\\ &= c(q)\left(\|(\text{curl } \overline{v})_\theta(\cdot,t)\|_{L^q(C_{1,4,1})} + \|\overline{v}(\cdot,t)\|_{L^q(C_{1,4,1})}\right).\end{aligned}$$

Note $(\text{curl } v)_\theta = \dfrac{\partial v_r}{\partial z} - \dfrac{\partial v_z}{\partial r} = (\text{curl } \overline{v})_\theta$, and so:

$$\|\nabla \overline{v}(\cdot,t)\|_{L^q(C_{2,3,1})} \le c(q)\left(\|(\text{curl } v)_\theta(\cdot,t)\|_{L^q(C_{1,4,1})} + \|v(\cdot,t)\|_{L^q(C_{1,4,1})}\right).$$

Thus,

$$\begin{aligned}\|\nabla v_r(\cdot,t)\|_{L^q(C_{2,3,1})} + \|\nabla v_z(\cdot,t)\|_{L^q(C_{2,3,1})} + \left\|\frac{v_r(\cdot,t)}{r}\right\|_{L^q(C_{2,3,1})}&\\ \le c(q)\left(\|(\text{curl } v)_\theta(\cdot,t)\|_{L^q(C_{1,4,1})} + \|v(\cdot,t)\|_{L^q(C_{1,4,1})}\right).&\end{aligned} \tag{4.3}$$

Here, $\left\|\dfrac{v_r(\cdot,t)}{r}\right\|_{L^q(C_{2,3,1})}$ is bounded due to the inequality:

$$\left\|\frac{v_r(\cdot,t)}{r}\right\|_{L^q(C_{2,3,1})} \le \left\|\frac{\partial v_r(\cdot,t)}{\partial r}\right\|_{L^q(C_{2,3,1})} + \left\|\frac{\partial v_z(\cdot,t)}{\partial z}\right\|_{L^q(C_{2,3,1})},$$

which comes from the divergence free equation. Taking the q-th power on (4.3) and integrating in time, we deduce the lemma. □

Taking $q = \frac{10}{3}$ in Lemma 4.2 yields the following Proposition:

Proposition 4.1. *For v, a smooth, axis symmetric solution to the Navier-Stokes equations in $Q_{1,4}$, then there exists a constant $c_1 > 0$ such that:*

$$\begin{aligned}\|\nabla v_r\|_{L^{\frac{10}{3}}(Q_{2,3})} + \left\|\frac{v_r}{r}\right\|_{L^{\frac{10}{3}}(Q_{2,3})} + \|\nabla v_z\|_{L^{\frac{10}{3}}(Q_{2,3})}&\\ \le c_1\left(\|\omega_\theta\|_{L^{\frac{10}{3}}(Q_{1,4})} + \|v\|_{L^{\frac{10}{3}}(Q_{1,4})}\right).&\end{aligned}$$

The right hand side is a priori bounded due to standard energy estimates and our Theorem 1.1 (i).

Proof of Theorem 1.1 (ii):
We use the scaling invariance of (4.1) and do the analysis in $P_{1,4,1} \subset Q_{1,4}$. We let V be the matrix:

$$V = \begin{bmatrix} \frac{\partial v_r}{\partial r} - \frac{1}{r^2} & \frac{\partial v_z}{\partial r} \\ \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix},$$

which can be regarded as a potential in the system when we take the equations together. Proposition 4.1 shows $V \in L^{\frac{10}{3}}(P_{1,4,1})$. This, along with our analysis on the drift term b as before implies, by a similar argument to that in Section 3, that ω_r and ω_z are also a priori bounded. Again, scaling, and in particular the scaling of $\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})}$, will come into play.

We let $q \geq 1$ be a rational number and choose $\psi = \phi(y)\eta(s)$ to be the same refined cut-off function as previously defined, satisfying the following:

$$\text{supp } \phi \subset C(\sigma_1);\ \phi(y) = 1 \text{ for all } \in C(\sigma_2);\ \frac{|\nabla\phi|}{\phi^\delta} \leq \frac{c_2}{\sigma_1 - \sigma_2} \text{ for } \delta \in (0,1),\ 0 \leq \phi \leq 1;$$

$$\text{supp } \eta \subset (-\sigma_1^2, 0];\ \eta(s) = 1 \text{ for all } s \in [-\sigma_2^2, 0];\ |\eta'| \leq \frac{c_3}{(\sigma_1 - \sigma_2)^2};\ 0 \leq \eta \leq 1.$$

We start by using $\omega_r^{2q-1}\psi^2$ as a test function on the first equation of system (4.1).

$$\begin{aligned} 0 &= \int_{P(\sigma_1)} \left(\Delta\omega_r - b \cdot \nabla\omega_r + \omega_r \left(\frac{\partial v_r}{\partial r} - \frac{1}{r^2} \right) + \omega_z \frac{\partial v_r}{\partial_z} - \frac{\partial \omega_r}{\partial s} \right) \omega_r^{2q-1}\psi^2 dyds \\ &= \int_{P(\sigma_1)} \omega_r^{2q-1}\psi^2 \Delta\omega_r dyds \\ &\qquad - \int_{P(\sigma_1)} \frac{1}{q} b \cdot \nabla(\omega_r^q)(\omega_r^q\psi^2) dyds - \int_{P(\sigma_1)} \frac{1}{q} \partial_s(\omega_r^q)(\omega_r^q\psi^2) dyds \\ &\qquad\qquad + \int_{P(\sigma_1)} \left(\frac{\partial v_r}{\partial r} - \frac{1}{r^2} \right)(\omega_r^{2q}\psi^2) + \left(\frac{\partial v_r}{\partial z} \right) \omega_z \omega_r^{2q-1}\psi^2 dyds. \end{aligned}$$

We work on the first term on the right hand side, using integration by parts, as usual, direct calculations, and algebraic manipulations:

$$\begin{aligned}
\int_{P(\sigma_1)} \omega_r^{2q-1}\psi^2\Delta\omega_r dyds &= -\int_{P(\sigma_1)} \nabla(\omega_r^{2q-1}\psi^2)\cdot\nabla\omega_r dyds \\
&= -\int_{P(\sigma_1)} (2q-1)(\omega_r^{2q-2}\nabla\omega_r)\cdot\nabla\omega_r\psi^2 + \omega^{2q-1}\nabla\omega_r\cdot\nabla(\psi^2) dyds \\
&= -\int_{P(\sigma_1)} (2q-1)(\omega_r^{q-1}\nabla\omega_r)\cdot(\omega_r^{q-1}\nabla\omega_r)\psi^2 + \nabla(\psi^2)\omega_r^q(\omega_r^{q-1}\nabla\omega_r) dyds \\
&= -\frac{2q-1}{q^2}\int_{P(\sigma_1)} \nabla(\omega_r^q)\cdot\nabla(\omega_r^q)\psi^2 dyds - \frac{1}{q}\int_{P(\sigma_1)} \omega_r^q\nabla(\omega_r^q)\cdot\nabla(\psi^2)dyds \\
&\le -\frac{1}{q}\int_{P(\sigma_1)} \nabla(\omega_r^q)\cdot\left(\nabla(\omega_r^q)\psi^2 + \nabla(\psi^2)\omega_r^q\right) dyds, \qquad \text{since } \frac{1}{q} < \frac{2q-1}{q^2}, \\
&= -\frac{1}{q}\int_{P(\sigma_1)} \nabla(\omega_r^q)\cdot\nabla(\omega_r^q\psi^2)dyds \\
&= -\frac{1}{q}\int_{P(\sigma_1)} \left(|\nabla(\omega_r^q\psi)|^2 - |\nabla\psi|^2\omega_r^{2q}\right) dyds.
\end{aligned}$$

This implies:

$$\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(\omega_r^q\psi)|^2 dyds \\
&\quad\le -\int_{P(\sigma_1)} b\cdot\nabla(\omega_r^q)(\omega_r^q\psi^2)dyds - \int_{P(\sigma_1)} \partial_s(\omega_r^q)(\omega_r^q\psi^2)dyds + \int_{P(\sigma_1)} |\nabla\psi|^2\omega_r^{2q} dyds \\
&\qquad + q\int_{P(\sigma_1)} \left[\left(\frac{\partial v_r}{\partial r} - \frac{1}{r^2}\right)(\omega_r^{2q}\psi^2) + \left(\frac{\partial v_r}{\partial z}\right)\omega_z\omega_r^{2q-1}\psi^2\right] dyds.
\end{aligned} \tag{4.4}$$

Similarly, we use $\omega_z^{2q-1}\psi^2$ as a test function in the second equation in system (4.1) to arrive at:

$$\begin{aligned}
&\int_{P(\sigma_1)} |\nabla(\omega_z^q\psi)|^2 dyds \\
&\quad\le -\int_{P(\sigma_1)} b\cdot\nabla(\omega_z^q)(\omega_z^q\psi^2)dyds - \int_{P(\sigma_1)} \partial_s(\omega_z^q)(\omega_z^q\psi^2)dyds + \int_{P(\sigma_1)} |\nabla\psi|^2\omega_z^{2q} dyds \\
&\qquad + q\int_{P(\sigma_1)} \left[\left(\frac{\partial v_z}{\partial z}\right)(\omega_z^{2q}\psi^2) + \left(\frac{\partial v_z}{\partial r}\right)\omega_r\omega_z^{2q-1}\psi^2\right] dyds.
\end{aligned} \tag{4.5}$$

We let $f = |\omega_r|^q + |\omega_z|^q$ and V represent the matrix:

$$V = \begin{bmatrix} \frac{\partial v_r}{\partial r} - \frac{1}{r^2} & \frac{\partial v_z}{\partial r} \\ \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}.$$

We add (4.4) and (4.5) and apply Cauchy-Schwartz inequality to the term involving V to obtain:

$$\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds \le 2\int_{P(\sigma_1)} \left(-b\cdot\nabla f(f\psi^2) - \partial_s f(f\psi^2) + |\nabla\psi|^2 f^2 + qc_5|V|f^2\psi^2\right) dyds.$$

Here $|V|$ is the max norm of the matrix. We proceed just as in the end of Step 1 in Section 3 to reach:

$$\begin{aligned}\int_{P(\sigma_1)} &|\nabla(f\psi)|^2 dyds + \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\ &\le -\int_{P(\sigma_1)} 2b\cdot\nabla f(f\psi^2)dyds + 2\int_{P(\sigma_1)} (\eta\partial_s\eta + |\nabla\psi|^2)f^2 dyds \\ &\qquad + c_4 q\int_{P(\sigma_1)} |V|f^2\psi^2 dyds \\ &:= T_1 + T_2 + T_3.\end{aligned} \tag{4.6}$$

Terms T_1 and T_2 are in the same form as to T_1 and T_2 in (3.4) of Section 3. Therefore, they are treated in an identical manner as found there. We recall the estimates on those terms now(see (3.5) and (3.6)):

$$|T_1| \le \epsilon_1 c_5 K_b^{\frac{4}{3}}(C_{1,4,1})\int_{P(\sigma_1)} |\nabla(f\psi)|^2 dyds + \frac{c_6\epsilon_1^{-3}}{(\sigma_1-\sigma_2)^4}\int_{P(\sigma_1)} f^2 dyds \tag{4.7}$$

$$|T_2| \le \frac{c_7}{(\sigma_1-\sigma_2)^2}\int_{P(\sigma_1)} f^2 dyds. \tag{4.8}$$

We proceed to term T_3 involving the matrix constructed from the potential terms in system (4.1). We employ Hölder's inequality twice here:

$$\begin{aligned}T_3 &= c_4 q\int_{P(\sigma_1)} |V|(f\psi)^2 dyds \\ &\le c_4 q\left(\int_{P(\sigma_1)} |V|^{\frac{10}{3}} dyds\right)^{\frac{3}{10}}\left(\int_{P(\sigma_1)} \left((f\psi)^2\right)^{\frac{10}{7}} dyds\right)^{\frac{7}{10}} \\ &= c_4 q\|V\|_{L^{\frac{10}{3}}(P(\sigma_1))}\left(\int_{P(\sigma_1)} (f\psi)^{\frac{20}{7}} dyds\right)^{\frac{7}{10}} \\ &\le c_4 q\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})}\left(\int_{P(\sigma_1)} (f\psi)^{\frac{20}{7}-a}(f\psi)^a dyds\right)^{\frac{7}{10}} \qquad 0 < a < \frac{20}{7} \\ &= c_4 q\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})}\left(\int_{P(\sigma_1)} (f\psi)^{(\frac{20}{7}-a)p} dyds\right)^{\frac{7}{10p}}\left(\int_{P(\sigma_1)} (f\psi)^{ap'} dyds\right)^{\frac{7}{10p'}}\end{aligned}$$

for $1 < p, p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. If $\left(\frac{20}{7} - a\right)p = \frac{10}{3}$ and $ap' = 2$, then $p = \frac{14}{9}$ and $p' = \frac{14}{5}$ and we get:

$$T_3 \le c_4 q\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})}\left(\int_{P(\sigma_1)} (f\psi)^{\frac{10}{3}} dyds\right)^{\frac{9}{20}}\left(\int_{P(\sigma_1)} (f\psi)^2 dyds\right)^{\frac{1}{4}}.$$

We apply Young's inequality with exponents $\frac{4}{3}$ and 4:

$$
\begin{aligned}
T_3 \leq & \left[\left(\frac{4\epsilon_2}{3}\right)^{\frac{3}{4}}\left(\int_{P(\sigma_1)}(f\psi)^{\frac{10}{3}}dyds\right)^{\frac{9}{20}}\right] \times \\
& \qquad \left[c_4 q\left(\frac{4\epsilon_2}{3}\right)^{-\frac{3}{4}}\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})}\left(\int_{P(\sigma_1)}(f\psi)^2 dyds\right)^{\frac{1}{4}}\right] \\
\leq & \ \epsilon_2\left(\int_{P(\sigma_1)}((f\psi)^2)^{\frac{5}{3}}dyds\right)^{\frac{3}{5}} + c_8 q^4\epsilon_2^{-3}\|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})}\int_{P(\sigma_1)}(f\psi)^2 dyds \\
\leq & \ \epsilon_2\|(f\psi)^2\|_{L^{\frac{5}{3}}(P(\sigma_1))} + c_8 q^4\epsilon_2^{-3}\|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})}\int_{P(\sigma_1)} f^2 dyds.
\end{aligned} \tag{4.9}
$$

Note, $\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})}$ can be controlled as a result of Proposition 4.1.

At this time we utilize in (4.6) the estimates for T_1 (4.7), T_2 (4.8), and T_3 (4.9), which then becomes:

$$
\begin{aligned}
& \int_{P(\sigma_1)}|\nabla(f\psi)|^2 dyds + \frac{1}{2}\int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\
& \leq \epsilon_1 c_5 K_b^{\frac{4}{3}}(C_{1,4,1})\int_{P(\sigma_1)}|\nabla(f\psi)|^2 dyds + \frac{c_6\epsilon_1^{-3}}{(\sigma_1-\sigma_2)^4}\int_{P(\sigma_1)} f^2 dyds + \frac{c_7}{(\sigma_1-\sigma_2)^2}\int_{P(\sigma_1)} f^2 dyds \\
& \qquad + \epsilon_2\|(f\psi)^2\|_{L^{\frac{5}{3}}(P(\sigma_1))} + c_8 q^4\epsilon_2^{-3}\|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})}\int_{P(\sigma_1)} f^2 dyds.
\end{aligned}
$$

Choose

$$
\epsilon_1 = \frac{1}{2c_5 K_b^{\frac{4}{3}}(C_{1,4,1})}
$$

and absorb the appropriate term the left hand side. We arrive at:

$$
\begin{aligned}
& \int_{P(\sigma_1)}|\nabla(f\psi)|^2 dyds + \int_{C(\sigma_1)} f^2(y,0)\phi^2(y)dy \\
& \qquad \leq \frac{c_9 K_b^4(C_{1,4,1})}{(\sigma_1-\sigma_2)^4}\int_{P(\sigma_1)} f^2 dyds + \frac{c_{10}}{(\sigma_1-\sigma_2)^2}\int_{P(\sigma_1)} f^2 dyds \\
& \qquad\qquad + 2\epsilon_2\|(f\psi)^2\|_{L^{\frac{5}{3}}(P(\sigma_1))} + c_{11} q^4\|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})}\int_{P(\sigma_1)} f^2 dyds \\
& \qquad \leq 2\epsilon_2\|(f\psi)^2\|_{L^{\frac{5}{3}}(P(\sigma_1))} + \frac{c_{12} q^4}{(\sigma_1-\sigma_2)^4}\left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1\right)\int_{P(\sigma_1)} f^2 dyds,
\end{aligned} \tag{4.10}
$$

noting $0 < \sigma_1 - \sigma_2 < 1$ and $q = 1 + \frac{2}{n} > 1$.

Now, recall (3.11) in Moser's iteration in Section 3, which follows from Hölder's inequality, the Sobolev inequality, $n=3$, and properties of the cut-off function. We have:

$$\int_{P(\sigma_1)} (\psi f)^{\frac{10}{3}} dyds \leq c_{13} \left(\sup_{-1\leq s<0} \int_{C(\sigma_1)} (f(y,s)\phi(y))^2 dy \right)^{\frac{2}{3}} \int_{P(\sigma_1)} |\nabla (f\psi)|^2 dyds.$$

Apply estimate (4.10), as we did in Section 3, and take the $\frac{3}{5}$ power of both sides:

$$\begin{aligned}\|(\psi f)^2\|_{L^{\frac{5}{3}}(P(\sigma_1))} \leq \frac{c_{14} q^4}{(\sigma_1-\sigma_2)^4} \left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1 \right) \int_{P(\sigma_1)} f^2 dyds \\ + 2c_{15}\epsilon_2 \|(f\psi)^2\|_{L^{\frac{5}{3}}(P(\sigma_1))}.\end{aligned}$$

Choose

$$\epsilon_2 = \frac{1}{4c_{15}},$$

absorb the appropriate term to the left, take the $\frac{5}{3}$ power of both sides, use the cut-off function, and recall $f = |\omega_r|^q + |\omega_z|^q$. We get:

$$\int_{P(\sigma_2)} (|\omega_r|^q + |\omega_z|^q)^{2\gamma} \leq c_{16} \left[\frac{c_{17} q^4}{\tau^4} \left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1 \right) \int_{P(\sigma_1)} (|\omega_r|^q + |\omega_z|^q)^2 dyds \right]^{\gamma},$$

where $\gamma = 1 + \frac{2}{n}$, $n=3$, $\tau = \sigma_1 - \sigma_2$. Define $h(x,t) = \max(|\omega_r|, |\omega_z|)$ and observe $h^q \leq |\omega_r|^q + |\omega_z|^q \leq 2h^q$. And so:

$$\int_{P(\sigma_2)} h^{2q\gamma} dyds \leq c_{16} \left[\frac{c_{18} q^4}{\tau^4} \left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1 \right) \int_{P(\sigma_1)} h^{2q} dyds \right]^{\gamma}. \quad (4.11)$$

Let $\tau_i = 2^{-i-2}$, $\sigma_0 = 1$, $\sigma_i = \sigma_{i-1} - \tau_i = 1 - \sum_{j=1}^{i} \tau_j$, $q = \gamma^i$. Thus we have an analogue to (3.6):

$$\int_{P(\sigma_{i+1})} h^{2\gamma^{i+1}} dyds \leq c_{16} \left[c_{19}^{i+2} \gamma^{4i} \left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1 \right) \int_{P(\sigma_1)} h^{2\gamma^i} dyds \right]^{\gamma}. \quad (4.12)$$

Raising both sides to the $\frac{1}{\gamma}$-th power, we get:

$$\left(\int_{P(\sigma_{i+1})} h^{2\gamma^{i+1}} dyds \right)^{\frac{1}{\gamma}} \leq c_{16}^{\frac{1}{\gamma}} \left[c_{19}^{i+2} \gamma^{4i} \left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1 \right) \int_{P(\sigma_1)} h^{2\gamma^i} dyds \right].$$

Now we apply (4.12) to the integral on the right hand side, with i replaced with $i-1$, to obtain:

$$\begin{aligned}\left(\int_{P(\sigma_{i+1})} h^{2\gamma^{i+1}} dyds \right)^{\frac{1}{\gamma}} \leq c_{16}^{\frac{1}{\gamma}} \left[c_{19}^{i+2} \gamma^{4i} \left(K_b^4(C1,4) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1 \right) \right] \times \\ 2c_{16} \left[c_{19}^{i+2} \gamma^{4i} \left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1 \right) \int_{P(\sigma_{i-1})} h^{2\gamma^{i-1}} dyds \right]^{\gamma}.\end{aligned}$$

Repeat this process and we arrive at:

$$\left(\int_{P(\sigma_{i+1})} h^{2\gamma^{i+1}} dyds\right)^{\frac{1}{\gamma^{i+1}}} \leq (2c_{16})^{\sum \gamma^{-j}} c_{19}^{\sum (j+1)\gamma^{-j+1}} \gamma^{4\sum (j-1)\gamma^{-j+1}} \times$$
$$\left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1\right)^{\sum \gamma^{-j+1}} \int_{P_{1,4,1}} h^2 dyds.$$

Note the sums in the exponents are all from $j = 1$ to $j = i+1$. Let $i \to \infty$. All the exponent series converge. In particular, the series in the exponent for $\left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1\right)$ converges to $\frac{5}{2}$. Note also that $\sigma_i \to \frac{3}{4}$. Therefore, we arrive at:

$$\sup_{P_{2,3,1}} \left(\omega_r^2 + \omega_z^2\right) \leq c_{20} \left(K_b^4(C_{1,4,1}) + \|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} + 1\right)^{\frac{5}{2}} \left(\int_{P_{1,4,1}} \omega_r^2 dyds + \int_{P_{1,4,1}} \omega_z^2 dyds\right). \tag{4.13}$$

It is time to note how $\|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})}$ is controlled. Recall:

$$V = \begin{bmatrix} \frac{\partial v_r}{\partial r} - \frac{1}{r^2} & \frac{\partial v_z}{\partial r} \\ \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}.$$

Applying Proposition 4.1 with $P_{1,4,1}$ being the domain on the left, $P_{\frac{1}{2},\frac{16}{3},1}$ being the domain on the right we can deduce that:

$$\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})} \leq c_{21} \left(\|\omega_\theta\|_{L^{\frac{10}{3}}(P_{\frac{1}{2},\frac{16}{3},1})} + \|v\|_{L^{\frac{10}{3}}(P_{\frac{1}{2},\frac{16}{3},1})} + 1\right) \tag{4.14}$$

Even though at this point we already know that V is a priori bounded by standard energy estimates and our pointwise bound on ω_θ, we use the method in Section 3 to prove a bound for $\|\omega_\theta\|_{L^{\frac{10}{3}}(P_{\frac{1}{2},\frac{16}{3},1})}$. This allows for better control of $\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})}$. The argument amounts to running Moser's iteration only once. Recall:

$$\Omega = \frac{\omega_\theta}{r}$$

and that in Section 3 we defined a constant Λ and functions:

$$\overline{\Omega}_+ = \begin{cases} \Omega + \Lambda & \Omega \geq 0, \\ \Lambda & \Omega < 0, \end{cases} \quad \overline{\Omega}_- = \begin{cases} -\Omega + \Lambda & \Omega \leq 0, \\ \Lambda & \Omega > 0. \end{cases} .$$

We will utilize estimate (3.12) to control the$L^{\frac{10}{3}}$ norm of ω_θ, but first we must manipulate the domains that appear in the inequality to fit our current setting. We recall (3.12) from Section 3:

$$\int_{P(\sigma_2)} \overline{\Omega}_+^{2q\gamma} dyds \leq c_{22} \left(\frac{c_{23} q^2}{\tau^4} \left(K_b^4(C_{1,4,1}) + \Lambda^2 + 1\right) \int_{P(\sigma_1)} \overline{\Omega}_+^{2q} dyds\right)^\gamma,$$

where $P(\sigma_i) = P_{5-4\sigma_i, 4\sigma_i, 1}$, $\tau = \sigma_1 - \sigma_2$, and $\gamma = 1 = \frac{2}{n}$. We replace this $P(\sigma_i)$ with $P(\sigma_i) = P_{\frac{1}{4}(5-4\sigma_i), \frac{64}{9}\sigma_i, 1}$. The argument over this domain would be identical to that in Section 3, with $\Lambda = \|v_\theta\|_{L^\infty(P_{\frac{1}{4},\frac{64}{9},1})} \leq 4\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}$, up until the point where we derive

(3.12). We recall the condition on q is $q \geq 1$ and the condition on σ_1, σ_2 here, in this setting, would be $\frac{265}{512} \leq \sigma_2 < \sigma_1 \leq 1$. Also note that $\gamma = 1 + \frac{2}{n},\ n = 3$ and so $\gamma = \frac{5}{3}$. We choose $q = 1,\ \sigma_1 = 1,\ \sigma_2 = \frac{3}{4}$ to get:

$$\int_{P_{\frac{1}{2},\frac{16}{3},1}} \overline{\Omega}_+^{\frac{10}{3}} dyds \leq c_{24} \left(\left(K_b^4(P_{\frac{1}{4},\frac{64}{9},1}) + \Lambda^2 + 1 \right) \int_{P_{\frac{1}{4},\frac{64}{9},1}} \overline{\Omega}_+^2 dyds \right)^{\frac{5}{3}}.$$

Similarly we can also get:

$$\int_{P_{\frac{1}{2},\frac{16}{3},1}} \overline{\Omega}_-^{\frac{10}{3}} dyds \leq c_{24} \left(\left(K_b^4(C_{\frac{1}{4},\frac{64}{9},1}) + \Lambda^2 + 1 \right) \int_{P_{\frac{1}{4},\frac{64}{9},1}} \overline{\Omega}_-^2 dyds \right)^{\frac{5}{3}}.$$

Taking the $\frac{3}{10}$ power of both sides we derive:

$$\|\overline{\Omega}_+\|_{L^{\frac{10}{3}}(P_{\frac{1}{2},\frac{16}{3},1})} \leq c_{25} \left(K_b^4(C_{\frac{1}{4},\frac{64}{9},1}) + \Lambda^2 + 1 \right)^{\frac{1}{2}} \|\overline{\Omega}_+\|_{L^2(P_{\frac{1}{4},\frac{64}{9},1})},$$

and

$$\|\overline{\Omega}_-\|_{L^{\frac{10}{3}}(P_{\frac{1}{2},\frac{16}{3},1})} \leq c_{25} \left(K_b^4(C_{\frac{1}{4},\frac{64}{9},1}) + \Lambda^2 + 1 \right)^{\frac{1}{2}} \|\overline{\Omega}_-\|_{L^2(P_{\frac{1}{4},\frac{64}{9},1})}.$$

We can combine the above two estimates to get:

$$\left\| \frac{\omega_\theta}{r} \right\|_{L^{\frac{10}{3}}(P_{\frac{1}{2},\frac{16}{3},1})} \leq c_{26} \left(K_b^4(C_{\frac{1}{4},\frac{64}{9},1}) + \Lambda^2 + 1 \right)^{\frac{1}{2}} \left\| \frac{\omega_\theta}{r} \right\|_{L^2(P_{\frac{1}{4},\frac{64}{9},1})}.$$

We note r is bounded between two positive constants on the left and on the right, to arrive at:

$$\|\omega_\theta\|_{L^{\frac{10}{3}}(P_{\frac{1}{2},\frac{16}{3},1})} \leq c_{27} \left(K_b^4(C_{\frac{1}{4},\frac{64}{9},1}) + \Lambda^2 + 1 \right)^{\frac{1}{2}} \|\omega_\theta\|_{L^2(P_{\frac{1}{4},\frac{64}{9},1})}.$$

Apply this to (4.14):

$$\|V\|_{L^{\frac{10}{3}}(P_{1,4,1})} \leq c_{28} \left(\left(K_b^4(C_{\frac{1}{4},\frac{64}{9},1}) + \Lambda^2 + 1 \right)^{\frac{1}{2}} \|\omega_\theta\|_{L^2(P_{\frac{1}{4},\frac{64}{9},1})} + \|v\|_{L^2(P_{\frac{1}{2},\frac{16}{3},1})} + 1 \right).$$

Thus,

$$\|V\|^4_{L^{\frac{10}{3}}(P_{1,4,1})} \leq c_{29} \left(\left(K_b^4(C_{\frac{1}{4},\frac{64}{9},1}) + \|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + 1 \right)^2 \|\omega_\theta\|^4_{L^2(P_{\frac{1}{4},\frac{64}{9},1})} + \|v\|^4_{L^2(P_{\frac{1}{2},\frac{16}{3},1})} + 1 \right),$$

utilizing $\Lambda \leq 4\|v_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}$. Apply this to (4.13), we get:

$$\sup_{P_{2,3,1}} \left(\omega_r^2 + \omega_z^2 \right) \leq A \left(\int_{P_{1,4,1}} \omega_r^2 dyds + \int_{P_{1,4,1}} \omega_z^2 dyds \right),$$

where A is the constant defined as:

$$A = c_{30} \left(K_b^4(C_{\frac{1}{10},10,1}) + \left(K_b^4(C_{\frac{1}{10},10,1}) + \|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + 1 \right) \|\omega_\theta\|^2_{L^2(P_{\frac{1}{10},10,1})} + \|v\|^2_{L^2(P_{\frac{1}{10},10,1})} + 1 \right)^5.$$

The domain is enlarged proportionally to make the right hand side more uniform.

Re-scaling: Recall our "tilde" notation and that what has actually been shown to this point is:

$$\sup_{P_{2,3,1}} \left(\widetilde{\omega}_r^2 + \widetilde{\omega}_z^2\right)(\widetilde{x}, \widetilde{t}) \le \widetilde{A}\left(\int_{P_{1,4,1}} \widetilde{\omega}_r^2 d\widetilde{x} d\widetilde{t} + \int_{P_{1,4,1}} \widetilde{\omega}_z^2 d\widetilde{x} d\widetilde{t}\right), \tag{4.15}$$

where $\widetilde{x} = \frac{x}{k}$, $\widetilde{t} = \frac{t}{k^2}$, $\widetilde{\omega}_r(\widetilde{x}, \widetilde{t}) = k^2 \omega_r(k\widetilde{x}, k^2\widetilde{t})$, $\widetilde{\omega}_z(\widetilde{x}, \widetilde{t}) = k^2 \omega_z(k\widetilde{x}, k^2\widetilde{t})$, and

$$\widetilde{A} = c_{30}\left(K_{\widetilde{b}}^4(C_{\frac{1}{10},10,1}) + \left(K_{\widetilde{b}}^4(C_{\frac{1}{10},10,1}) + \|\widetilde{r}\widetilde{v}_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + 1\right)\|\widetilde{\omega}_\theta\|^2_{L^2(P_{\frac{1}{10},10,1})} + \|\widetilde{v}\|^2_{L^2(P_{\frac{1}{10},10,1})} + 1\right)^5.$$

From the scaling in Section 2:

$$K_{\widetilde{b}}(C_{\frac{1}{10},10,1}) = \|\widetilde{b}(\widetilde{x}, \widetilde{t})\|_{L^\infty(-1,0;(C_{\frac{1}{10},10,1}))} = \frac{1}{k^{\frac{1}{2}}}\|b\|_{L^\infty(-k^2,0;L^2(C_{\frac{1}{10},10,k}))},$$

$$\|\widetilde{v}(\widetilde{x}, \widetilde{t})\|_{L^2(P_{\frac{1}{10},10,1})}\frac{1}{k^{\frac{3}{2}}} = \|v(x,t)\|_{L^2(P_{\frac{1}{10},10,k})},$$

and

$$\|\widetilde{\omega}(\widetilde{x}, \widetilde{t})\|_{L^2(P_{\frac{1}{10},10,1})} = \frac{1}{k^{\frac{1}{2}}}\|\omega(x,t)\|_{L^2(P_{\frac{1}{10},10,k})}.$$

Also $\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)}$ is scaling invariant. Finally, $\widetilde{A}$ scales in the following way:

$$\begin{aligned}\widetilde{A} &= c_{30}\left(K_{\widetilde{b}}^4(C_{\frac{1}{10},10,1}) + \left(K_{\widetilde{b}}^4(C_{\frac{1}{10},10,1}) + \|\widetilde{r}\widetilde{v}_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + 1\right)\|\widetilde{\omega}_\theta\|^2_{L^2(P_{\frac{1}{10},10,1})} + \|\widetilde{v}\|^2_{L^2(P_{\frac{1}{10},10,1})} + 1\right)^5 \\ &= \frac{c_{31}}{k^{15}}\Bigg[\left(K_b^4(C_{\frac{1}{10},10,k}) + k^2\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + k^2\right)\|\omega_\theta\|^2_{L^2(P_{\frac{1}{10},10,k})} \\ &\qquad + kK_b^4(C_{\frac{1}{10},10,k}) + \|v\|^2_{L^2(P_{\frac{1}{10},10,k})} + k^3\Bigg]^5.\end{aligned}$$

Apply all of this to (4.15) to achieve:

$$\begin{aligned}&\sup_{P_{2,3,k}} k^4\left(\omega_r^2(x,t) + \omega_z^2(x,t)\right) \\ &\quad \le \frac{c_{31}}{k^{16}}\Bigg[\left(K_b^4(C_{\frac{1}{10},10,k}) + k^2\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + k^2\right)\|\omega_\theta\|^2_{L^2(P_{\frac{1}{10},10,k})} \\ &\qquad kK_b^4(C_{\frac{1}{10},10,k}) + \|v\|^2_{L^2(P_{\frac{1}{10},10,k})} + k^3\Bigg]^5\left(\|\omega_r\|^2_{L^2(P_{\frac{1}{10},10,k})} + \|\omega_z\|^2_{L^2(P_{\frac{1}{10},10,k})}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\omega_r\|_{L^\infty(P_{2,3,k})} + \|\omega_z\|_{L^\infty(P_{2,3,k})} \\ &\quad \le \frac{c_{32}}{k^{10}}\Bigg[\left(K_b^4(C_{\frac{1}{10},10,k}) + k^2\|rv_{0,\theta}\|_{L^\infty(\mathbb{R}^3)} + k^2\right)\|\omega_\theta\|^2_{L^2(P_{\frac{1}{10},10,k})} \\ &\qquad + kK_b^4(C_{\frac{1}{10},10,k}) + \|v\|^2_{L^2(P_{\frac{1}{10},10,k})} + k^3\Bigg]^{\frac{5}{2}}\left(\|\omega_r\|_{L^2(P_{\frac{1}{10},10,k})} + \|\omega_z\|_{L^2(P_{\frac{1}{10},10,k})}\right).\end{aligned}$$

This proves (ii) of Theorem 1.1.

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